Singularity, Bifurcation and Reconfiguration of a Special Origami Pattern

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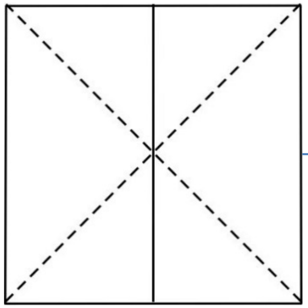
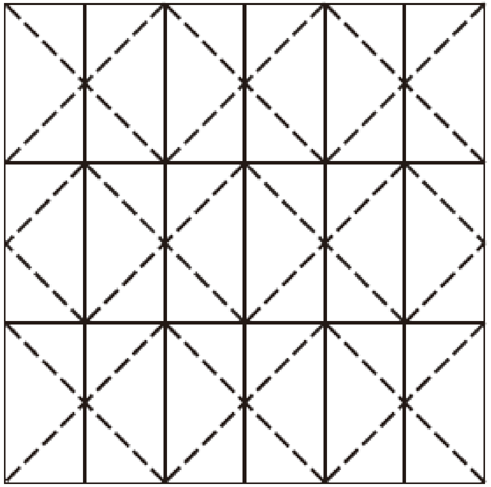
Outline

1. Introduction
2. Symmetric folding of the waterbomb origami pattern
3. Thick panel problem of the waterbomb pattern
4. Waterbomb origami tube
5. Perspectives on the waterbomb pattern

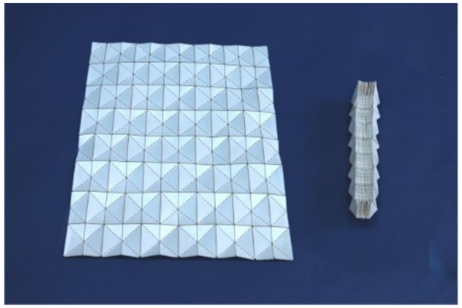
➢ Waterbomb pattern



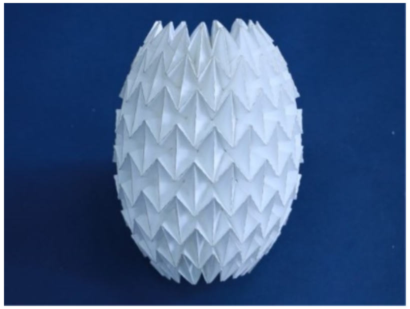
six-crease
waterbomb base



waterbomb tessellation



flat-foldable surface



deformable tube

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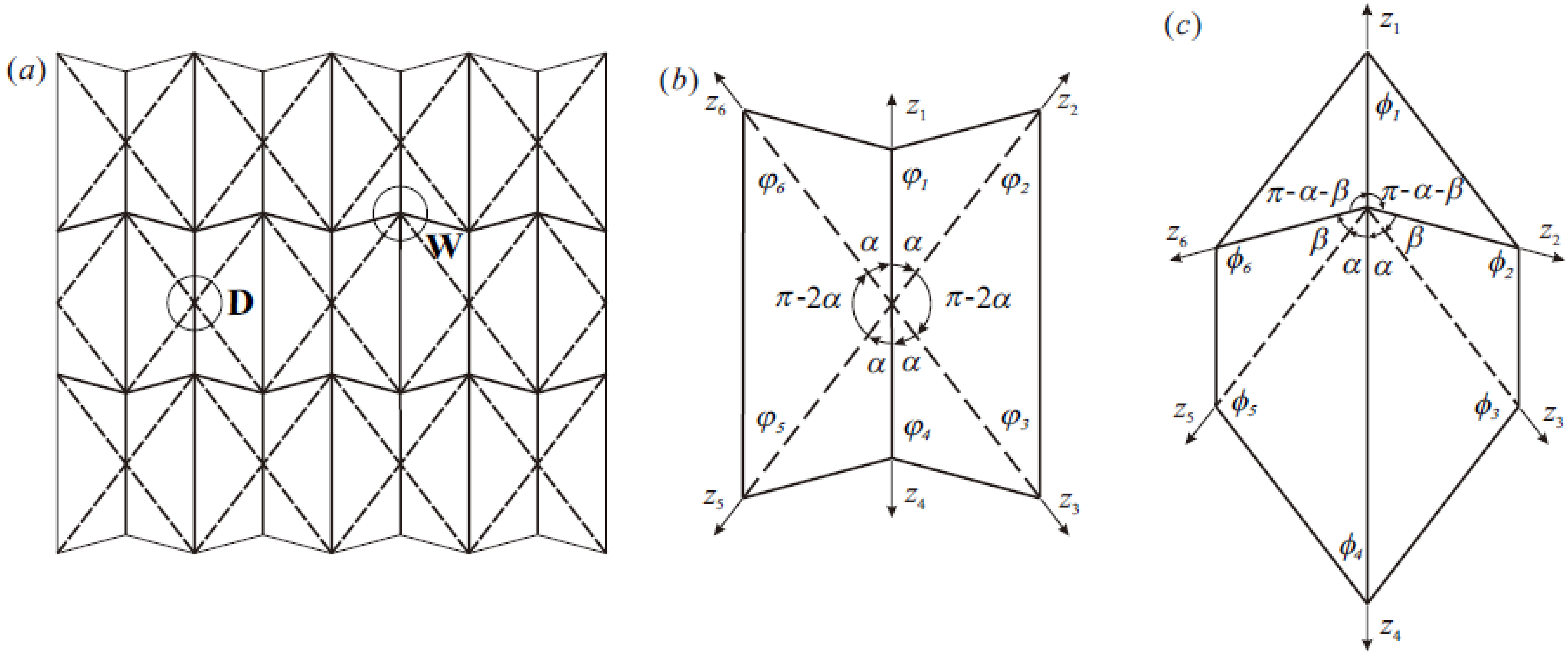
2. Symmetric folding of the waterbomb origami pattern



1) Unit **D** is a ***line- and plane-symmetric spherical 6R linkage*** with the geometric parameters

$$\alpha_{12} = \alpha_{34} = \alpha_{45} = \alpha_{61} = \alpha$$
$$\alpha_{23} = \alpha_{56} = \pi - 2\alpha$$

2) Unit **W** can also be considered as a ***plane-symmetric spherical 6R linkage*** with the geometric parameters

$$\alpha_{12} = \alpha_{61} = \pi - \alpha - \beta$$
$$\alpha_{23} = \alpha_{56} = \beta\,,\ \ \alpha_{34} = \alpha_{45} = \alpha$$

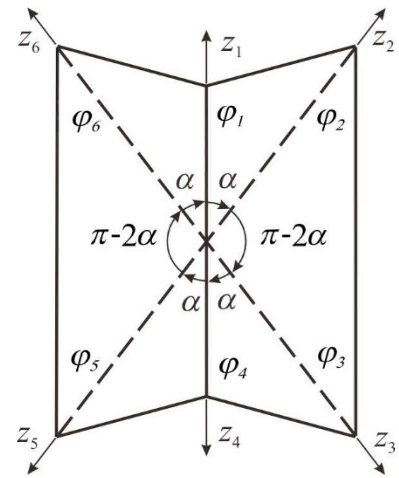
2. Symmetric folding of the waterbomb origami pattern

- Closure equations of the unit **D**:

$$\tan\frac{\varphi_1}{2}=\frac{1}{\cos\alpha}\cdot\tan\frac{\varphi_2}{2},\ \varphi_1=\varphi_4,\ \varphi_2=\varphi_3=\varphi_5=\varphi_6$$

- Compatibility condition: $\phi_1=\varphi_1,\ \phi_3=\varphi_2$

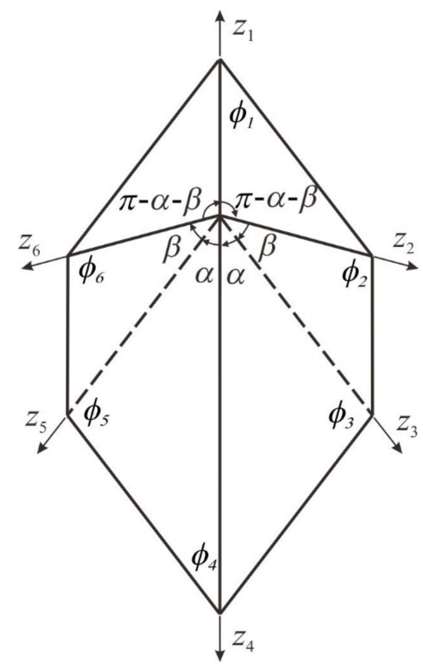
- Closure equations of the unit **W**:



1) $\tan\frac{\phi_1}{2}=\frac{1}{\cos\alpha}\tan\frac{\phi_3}{2},\ \tan\frac{\phi_2}{2}=\frac{\cos(\alpha+\beta)}{\cos\alpha}\tan\frac{\phi_3}{2},\ \phi_4=\phi_1,\ \phi_5=\phi_3,\ \phi_6=\phi_2$

2) $\tan\frac{\phi_1}{2}=\frac{1}{\cos\alpha}\tan\frac{\phi_3}{2},\ \tan\frac{\phi_2}{2}=\frac{\sin(\alpha+\beta)\tan^2\frac{\phi_3}{2}+\sin(\beta-\alpha)}{2\sin\alpha\tan\frac{\phi_3}{2}},\ \phi_5=\phi_3,\ \phi_6=\phi_2,$

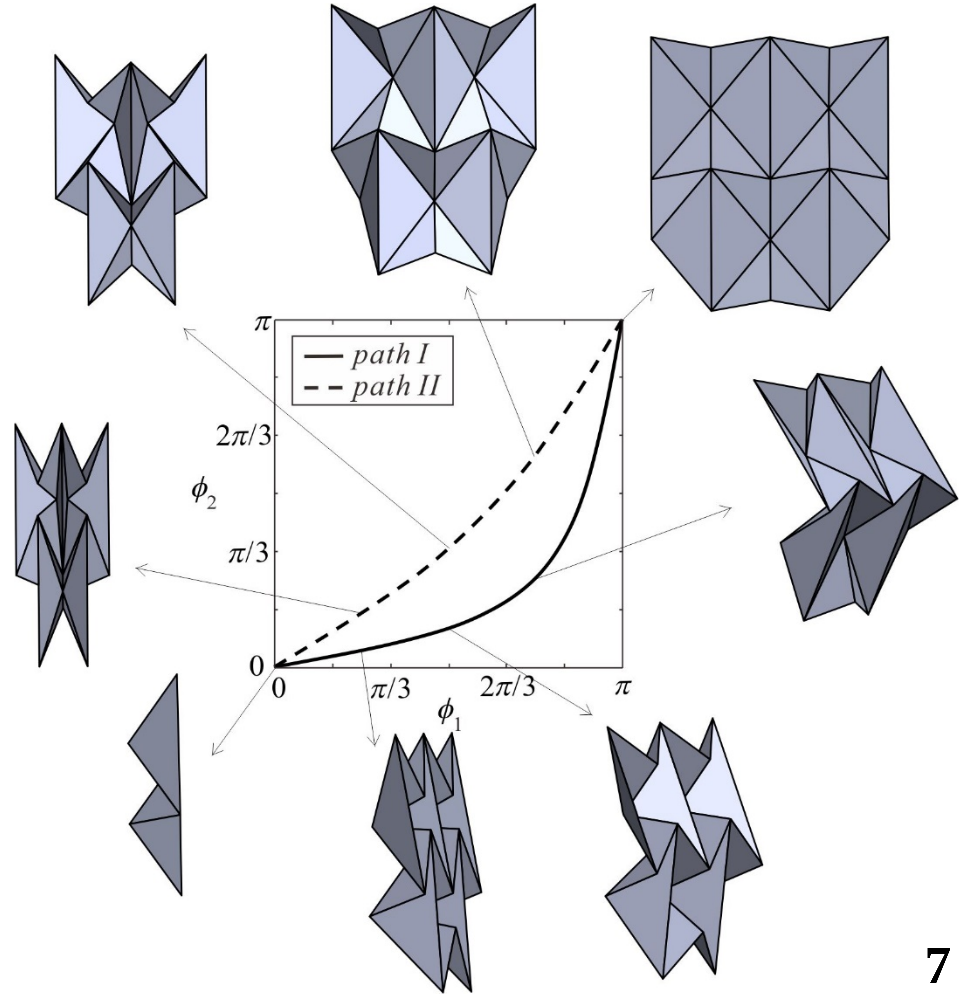
$$\tan\frac{\phi_4}{2}=\frac{\tan\frac{\phi_3}{2}\left(2\sin^2(\alpha+\beta)\tan^4\frac{\phi_3}{2}-4(\cos^2(\alpha+\beta)-\cos 2\beta)\tan^2\frac{\phi_3}{2}-2\sin(\beta-\alpha)(2\sin(\alpha+\beta)+\sin(\beta-\alpha))\right)}{\sin(\alpha+\beta)(7\sin\beta-\sin(2\alpha+\beta))\tan^4\frac{\phi_3}{2}+4(\sin\alpha\sin 2\beta+\cos\alpha\sin(\alpha+\beta)\sin(\beta-\alpha))\tan^2\frac{\phi_3}{2}-2\cos\alpha\sin^2(\beta-\alpha)}$$



2. Symmetric folding of the waterbomb origami pattern

➢ Behaviour of the waterbomb origami tessellation

(a) When $\alpha + \beta < \frac{\pi}{2}$ and $\alpha = \beta$, there are two smooth folding paths without either two-stage motion or blockage;



- Design angular parameters:

$$\alpha = \beta = \frac{2\pi}{9}$$

- Singular points:

$$(0, 0) \ , \ (\pi, \pi)$$

➢ Behaviour of the waterbomb origami tessellation

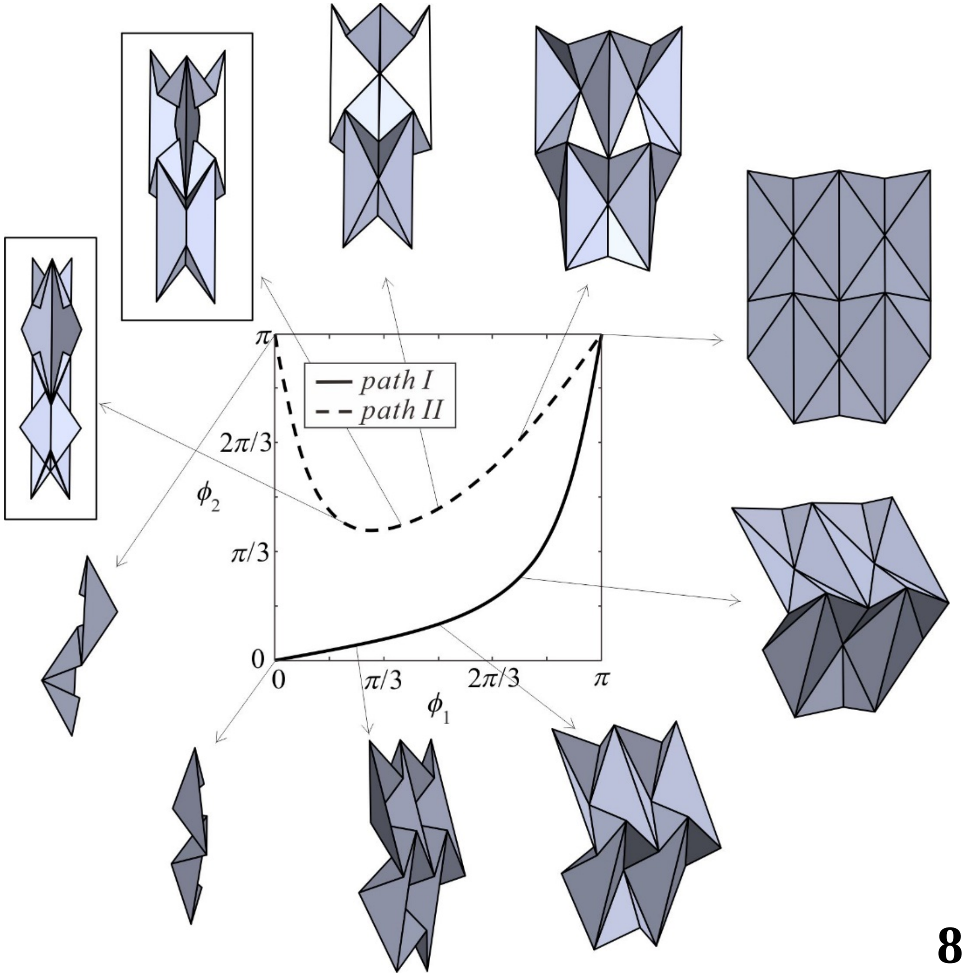
(b) When $\alpha + \beta < \frac{\pi}{2}$ and $\alpha \neq \beta$, *path II* is blocked and *path I* is smooth;

- Design angular parameters:

$$\alpha = \frac{7\pi}{36}, \ \beta = \frac{\pi}{4}$$

- Singular point:

$$(\pi, \pi)$$



2. Symmetric folding of the waterbomb origami pattern

➢ Behaviour of the waterbomb origami tessellation

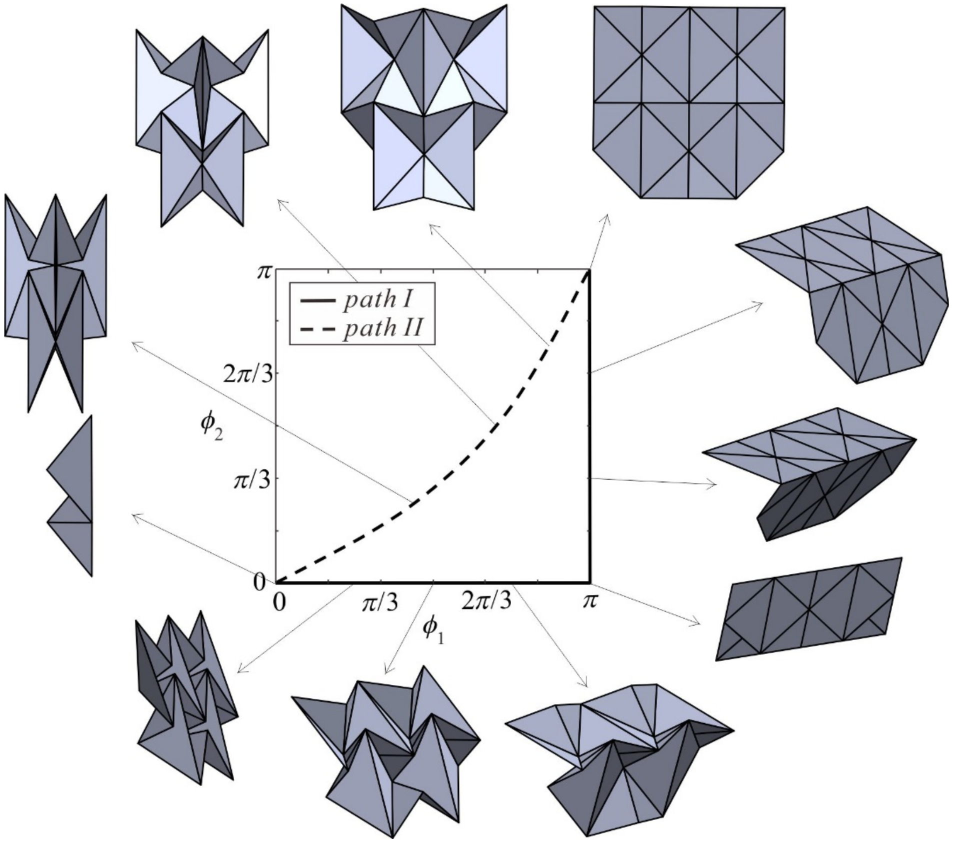
(c) When $\alpha + \beta = \frac{\pi}{2}$ and $\alpha = \beta$, *path I* is in two-stage motion, whereas *path II* is smooth;

- Design angular parameters:

$$\alpha = \beta = \frac{\pi}{4}$$

- Singular points:

$$(0, 0)\ ,\ (\pi, \pi)$$



➢ Behaviour of the waterbomb origami tessellation

(d) When $\alpha + \beta = \frac{\pi}{2}$ and $\alpha \neq \beta$, both two-stage motion on *path I* and blockage on *path II* happen;

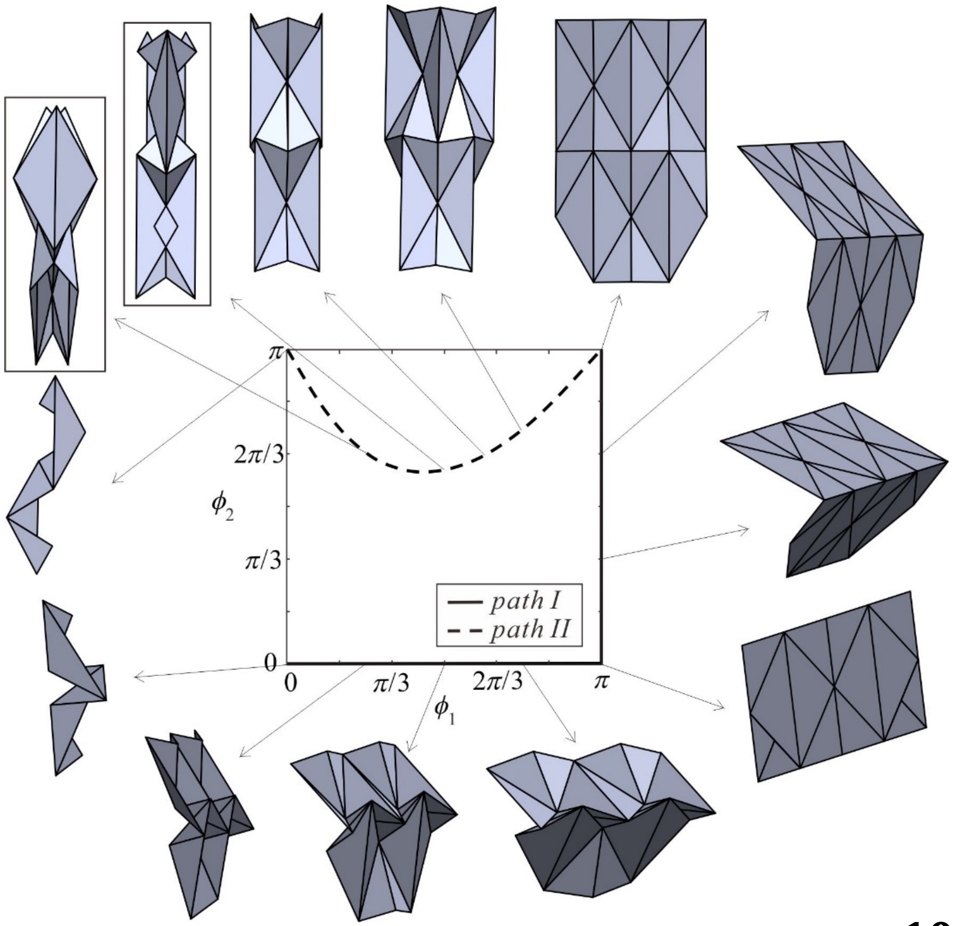
- Design angular parameters:

$$\alpha = \frac{\pi}{6}, \ \beta = \frac{\pi}{3}$$

- Singular points:

$$(\pi, \pi)$$

(e) When $\alpha + \beta > \frac{\pi}{2}$, both paths are blocked.



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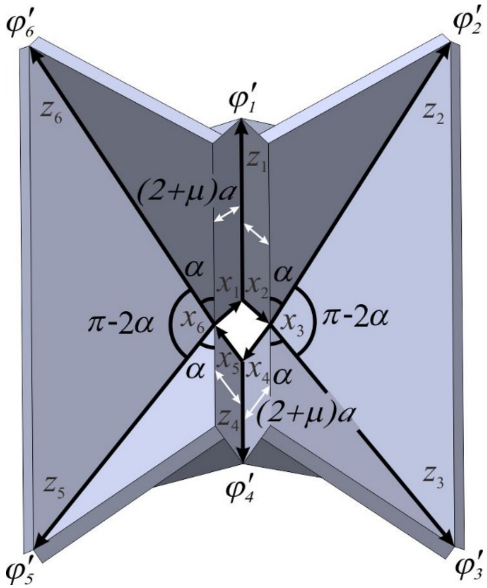
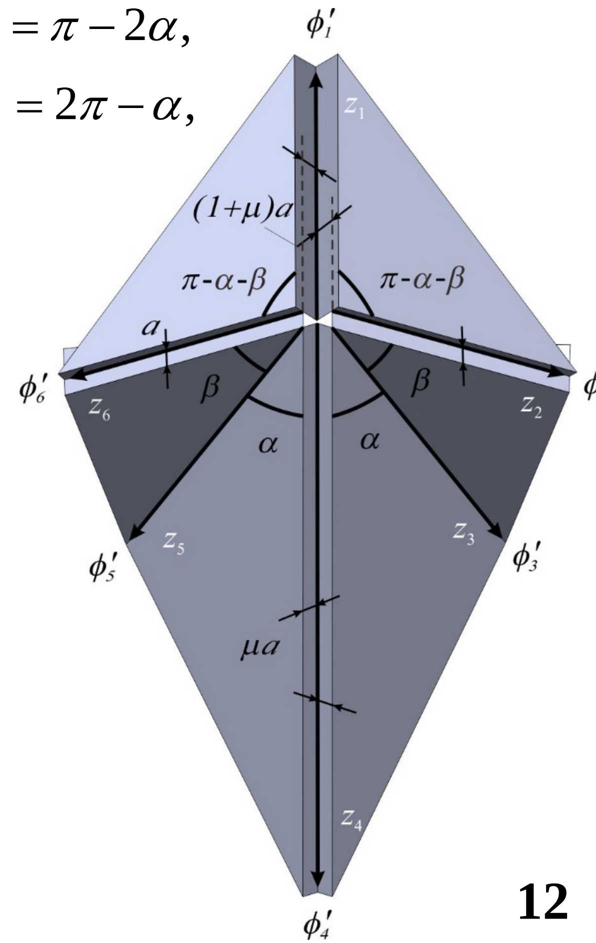
3. Thick panel problem of the waterbomb pattern



1) Unit **D** is a ***line- and plane- symmetric Bricard 6R linkage*** with the geometric parameters

$$a_{12}^D = a_{61}^D = a_{34}^D = a_{45}^D = (2+\mu)a,\ a_{23}^D = a_{56}^D = 0,$$
$$\alpha_{12}^D = 2\pi - \alpha,\ \alpha_{61}^D = \alpha,\ \alpha_{23}^D = \pi - 2\alpha,$$
$$\alpha_{56}^D = \pi + 2\alpha,\ \alpha_{34}^D = \alpha,\ \alpha_{45}^D = 2\pi - \alpha,$$
$$R_i^D = 0\ (i = 1, 2, 3, 4, 5, 6)$$



2) Unit **W** is a ***plane- symmetric Bricard 6R linkage*** with the geometric parameters

$$a_{12}^W = a_{61}^W = (1+\mu)a,\ a_{23}^W = a_{56}^W = a,\ a_{34}^W = a_{45}^W = \mu a,$$
$$\alpha_{12}^W = \pi - \alpha - \beta,\ \alpha_{61}^W = \pi + \alpha + \beta,\ \alpha_{23}^W = \beta,$$
$$\alpha_{56}^W = 2\pi - \beta,\ \alpha_{34}^W = 2\pi - \alpha,\ \alpha_{45}^W = \alpha,$$
$$R_i^W = 0\ (i = 1, 2, 3, 4, 5, 6)$$

3. Thick panel problem of the waterbomb pattern

- Closure equation of the unit **D**:

$$\tan\frac{\varphi_1'}{2}=\frac{1}{\cos\alpha}\tan\frac{\varphi_2'}{2},\ \varphi_4'=\varphi_1',\ \varphi_2'=\varphi_3'=\varphi_5'=\varphi_6'$$

- Closure equation of the unit **W**:

$$1)\ \tan\frac{\phi_1'}{2}=\frac{1}{\cos\alpha}\tan\frac{\phi_3'}{2},\ \tan\frac{\phi_2'}{2}=\frac{\cos(\alpha+\beta)}{\cos\alpha}\tan\frac{\phi_3'}{2},\ \phi_4'=\phi_1',\ \ \phi_5'=\phi_3',\ \ \phi_6'=\phi_2'$$

$$2)\ \tan\frac{\phi_1'}{2}=\frac{\tan\frac{\phi_3'}{2}\left(\mu\sin^2(\alpha+\beta)\tan^2\frac{\phi_3'}{2}+(\mu+1)(\mu\sin^2\beta+\sin^2\alpha)\right)}{\sin(\alpha+\beta)(\mu^2\sin\beta+\cos(\alpha+\beta)\sin\alpha)\tan^2\frac{\phi_3'}{2}+(\mu+1)^2\sin\alpha\sin\beta\cos\beta},$$

$$\tan\frac{\phi_2'}{2}=\frac{\mu\sin(\alpha+\beta)}{(\mu+1)\sin\alpha}\tan\frac{\phi_3'}{2},\ \phi_5'=\phi_3',\ \phi_6'=\phi_2',$$

$$\tan\frac{\phi_4'}{2}=\frac{\tan\frac{\phi_3'}{2}\left(4\mu\sin\alpha\sin^2(\alpha+\beta)\tan^2\frac{\phi_3'}{2}-4(\mu+1)\sin\alpha\left((\mu+1)\sin^2\beta-\sin^2(\alpha+\beta)\right)\right)}{\left(\begin{array}{l}\left(\cos(3\alpha+\beta)-2(1+\mu)^2\cos(\alpha+\beta)+(1+4\mu+2\mu^2)\cos(\alpha-\beta)\right)\sin(\alpha+\beta)\tan^2\frac{\phi_3'}{2}\\+2(\mu+1)^2\sin^2\alpha\sin2\beta\end{array}\right)}$$

3. Thick panel problem of the waterbomb pattern

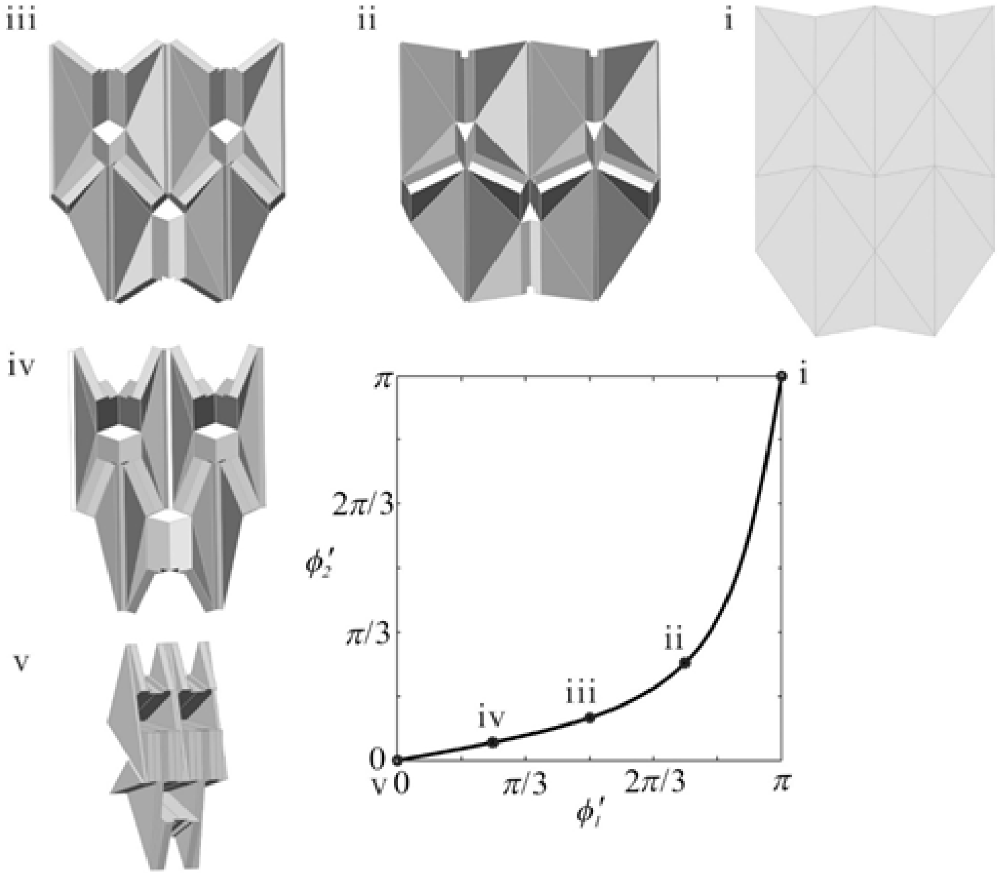
➢ Behaviour of the general thick-panel waterbomb

(a) For any $\mu \neq 0$, when $\alpha + \beta < \dfrac{\pi}{2}$, there is only one smooth folding path;

- Design parameters:

$$\alpha = \frac{7\pi}{36},\ \ \beta = \frac{\pi}{4},\ \ \mu = 0.5$$

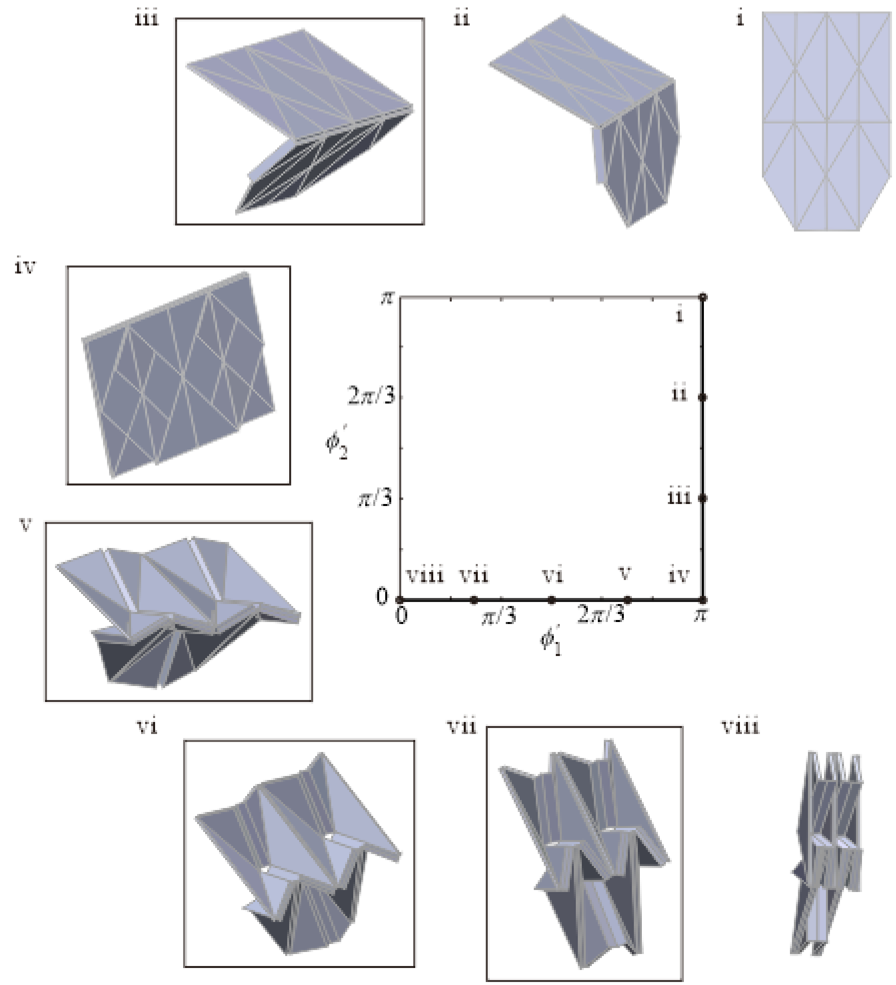
- Nonsingular



3. Thick panel problem of the waterbomb pattern

➢ Behaviour of the general thick-panel waterbomb

(b) For any $\mu \neq 0$, when $\alpha + \beta = \dfrac{\pi}{2}$, there is one two-stage folding path with blockage;



- Design parameters:

$$\alpha = \frac{\pi}{6},\ \ \beta = \frac{\pi}{3},\ \ \mu = 0.7$$

- Nonsingular

3. Thick panel problem of the waterbomb pattern

➢ Behaviour of the general thick-panel waterbomb

(c) For any $\mu \neq 0$, when $\alpha + \beta > \frac{\pi}{2}$, there is one blocked folding path;

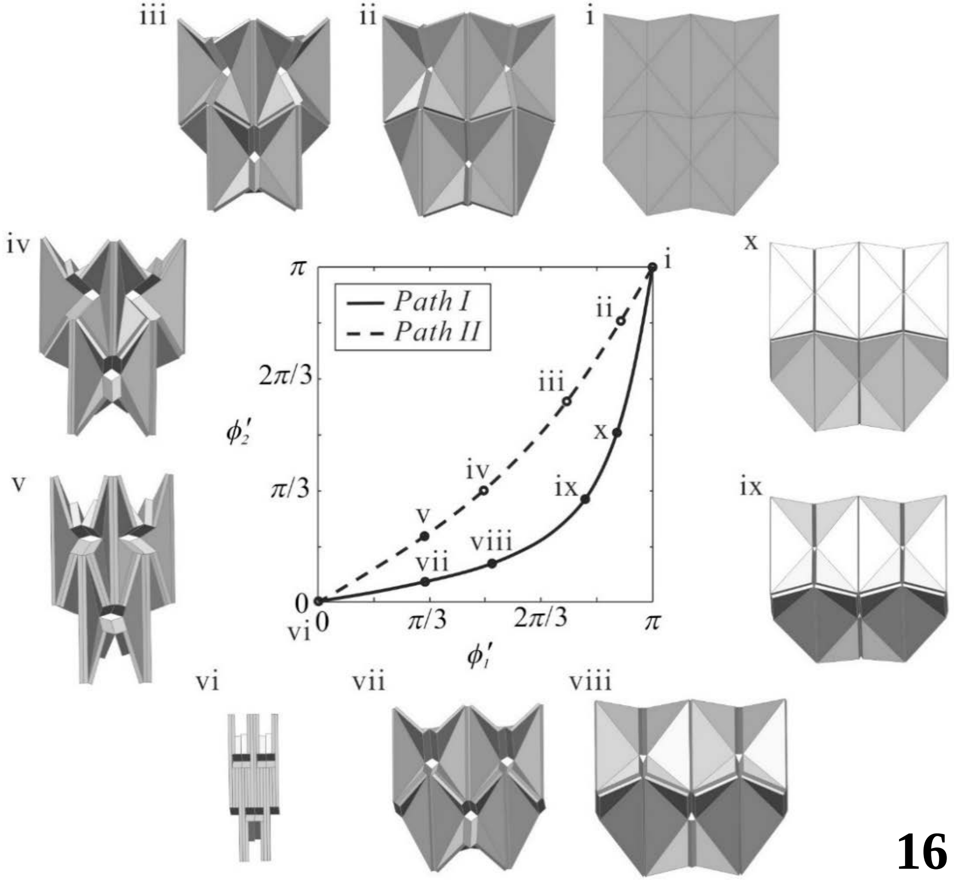
(d) For $\mu = 1$, when $\alpha + \beta < \frac{\pi}{2}$, $\alpha = \beta$, there are two smooth folding paths;

- Design parameters:

$$\alpha = \beta = \frac{2\pi}{9}, \quad \mu = 1$$

- Singular points:

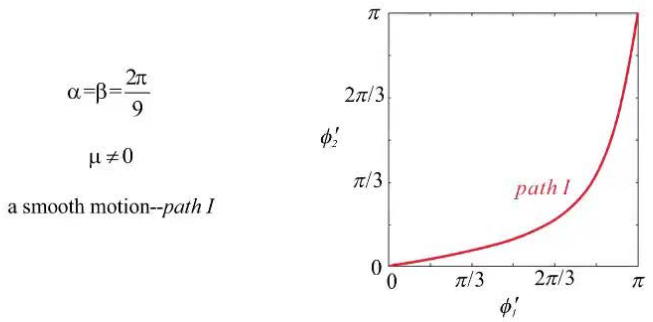
$$(0, 0)\ ,\ (\pi, \pi)$$

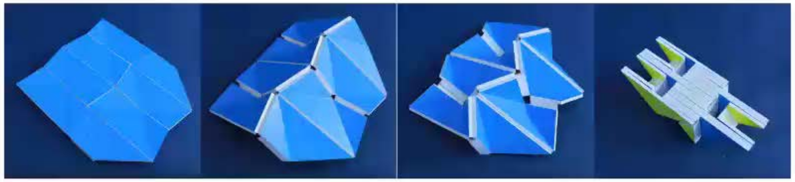


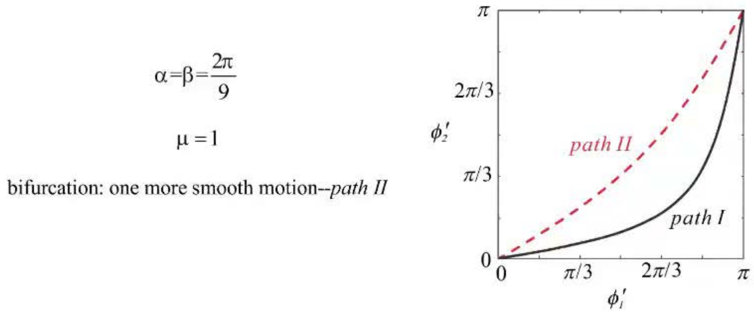
➢ Behaviour of the general thick-panel waterbomb

$$\alpha = \beta = \frac{2\pi}{9},\ \ \mu = 1$$

Path I







Path II

3. Thick panel problem of the waterbomb pattern

➢ Behaviour of the general thick-panel waterbomb

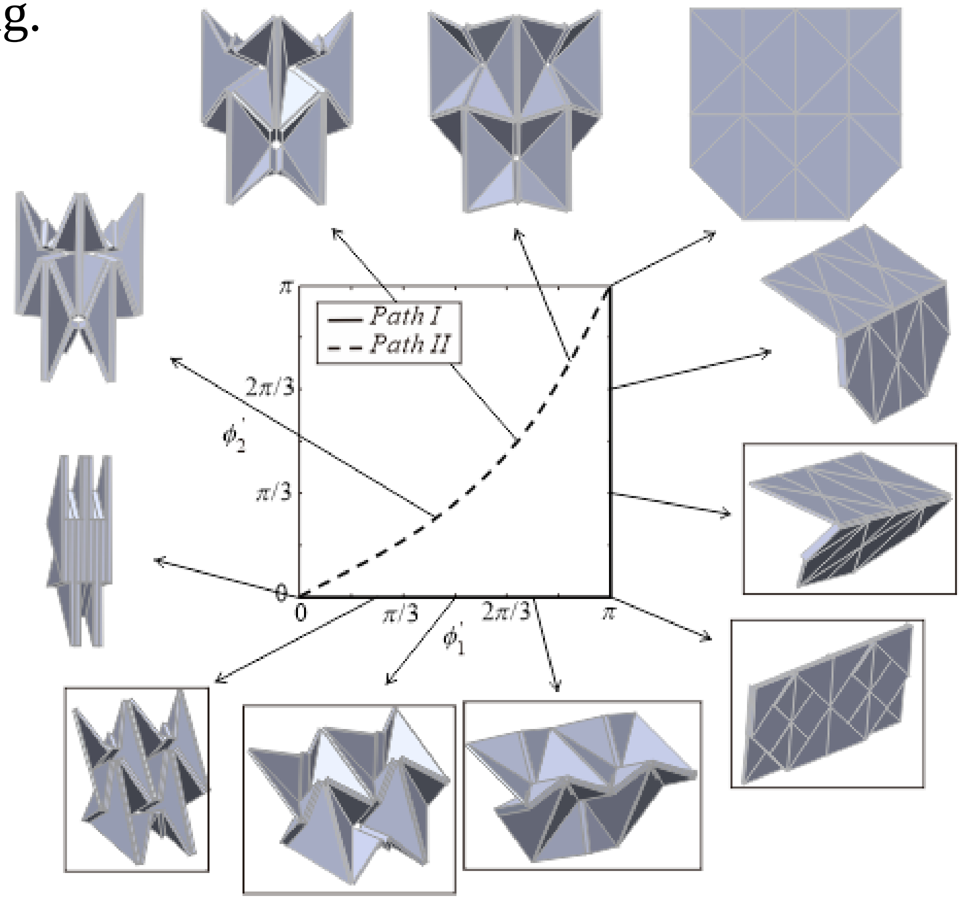
(e) For $\mu = 1$, when $\alpha = \beta = \dfrac{\pi}{4}$, Path *I* is in two-stage motion and blocked, but Path *II* can achieve smooth folding.

- Design parameters:

$$\alpha = \beta = \frac{\pi}{4}, \ \ \mu = 1$$

- Singular point:

$$(\pi, \pi)$$

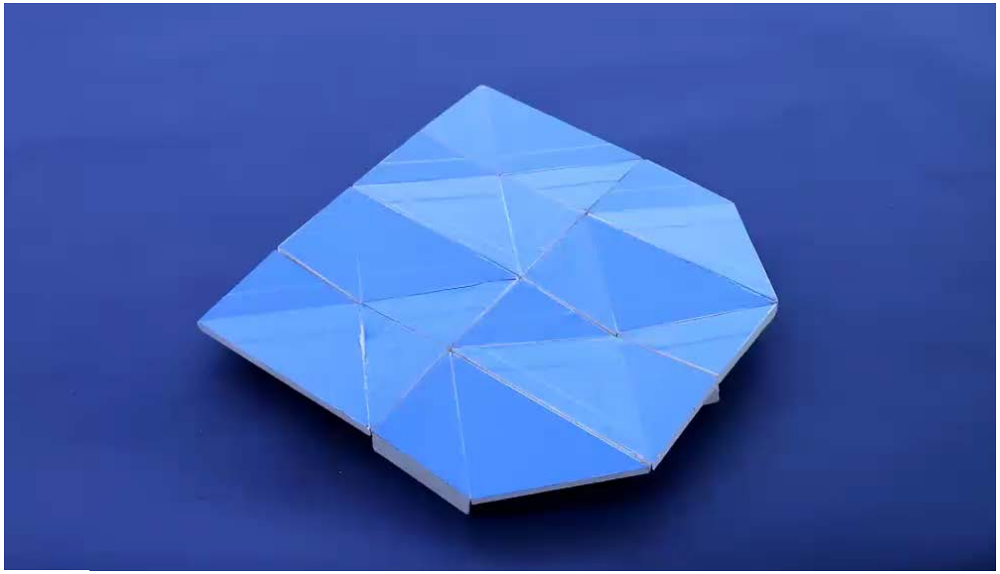


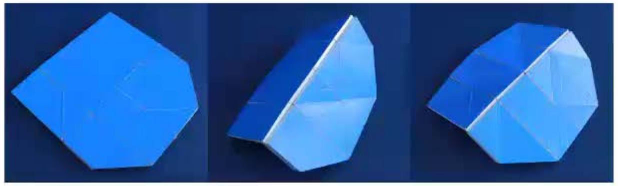
3. Thick panel problem of the waterbomb pattern

➢ **Behaviour of the general thick-panel waterbomb**

$$\alpha = \beta = \frac{\pi}{4}, \ \ \mu = 1$$

Path I

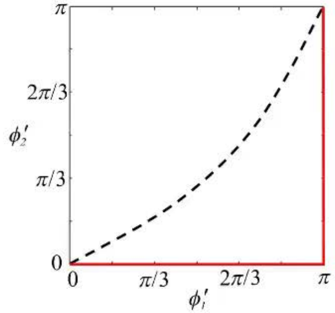




$\alpha = \beta = \frac{\pi}{4}$

$\mu = 1$

bifurcation: a two-stage motion and blocked



Path II

3. Thick panel problem of the waterbomb pattern

- **Summary**

- The rigid origami of the waterbomb tessellation of both zero-thickness sheet and thick panel under the symmetric motion condition are analyzed.
- The condition of bifurcation for both zero-thickness sheet and thick panel :

$$\alpha + \beta < \frac{\pi}{2}$$

$$\alpha = \beta$$

$$\mu = 1$$

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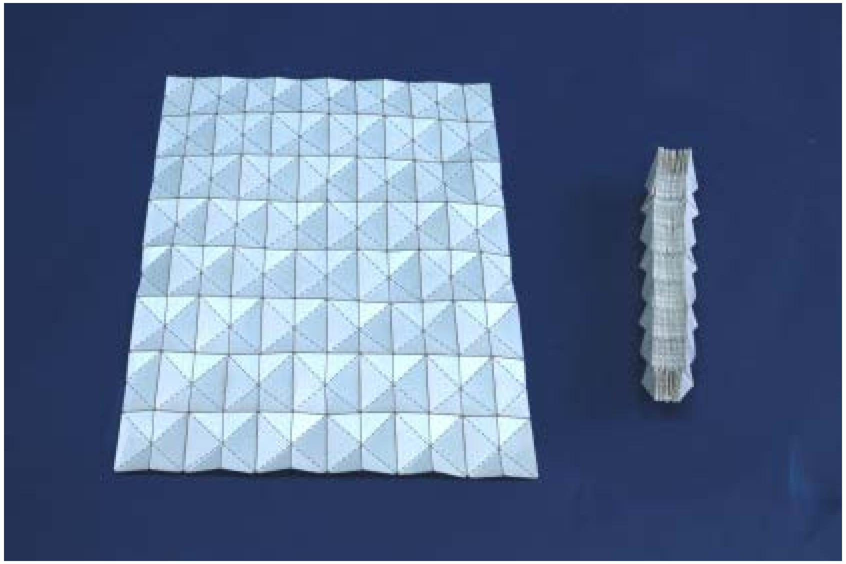
4. Waterbomb origami tube

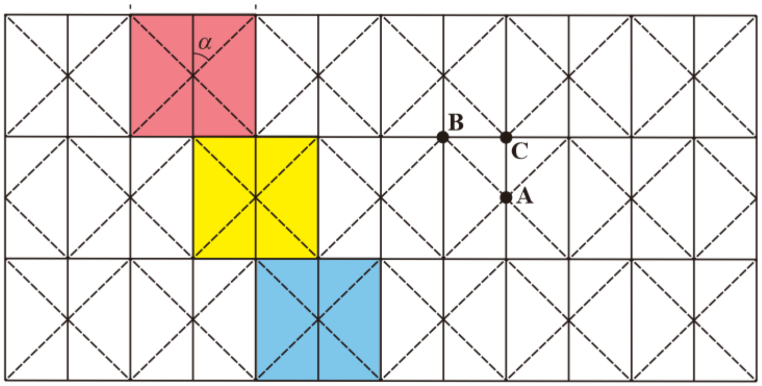
➢ Kinematics of the waterbomb tube



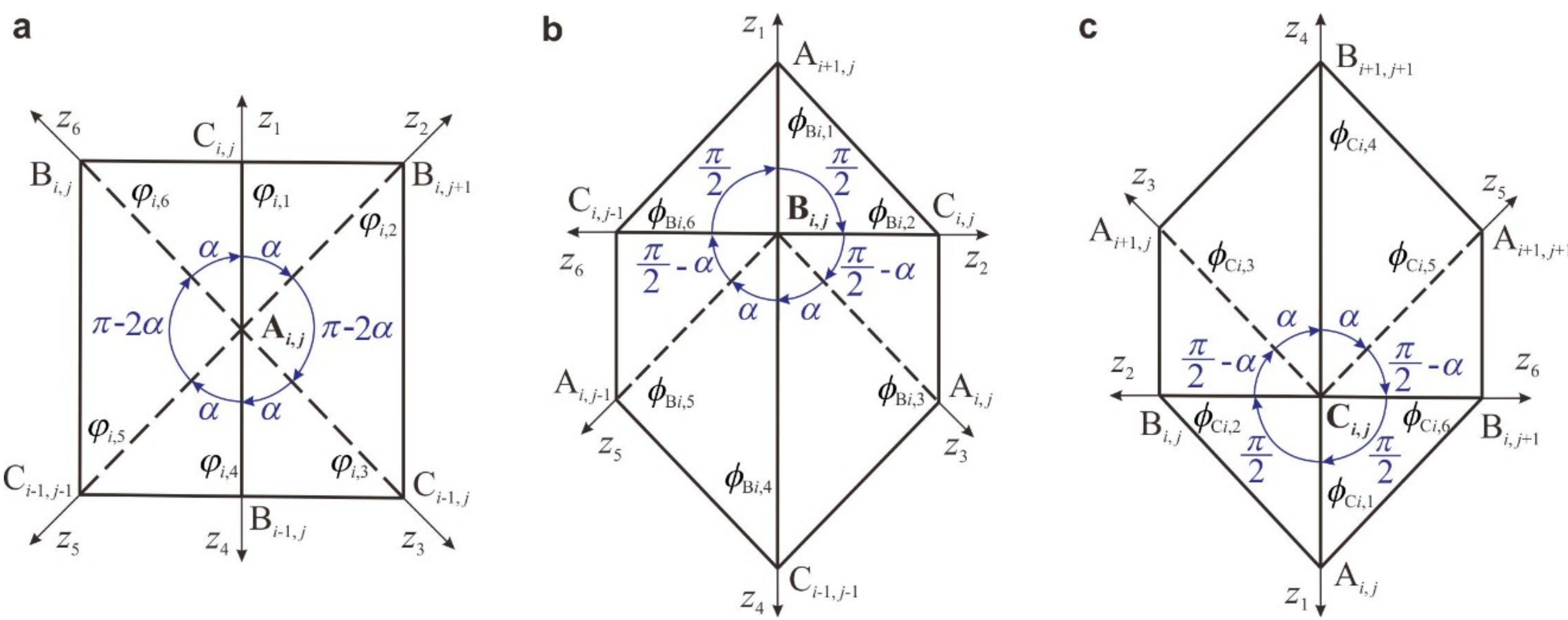
The motion of all waterbomb bases are the same

The motion of all waterbomb bases in the same layer are identical

➢ Kinematics of the waterbomb tube

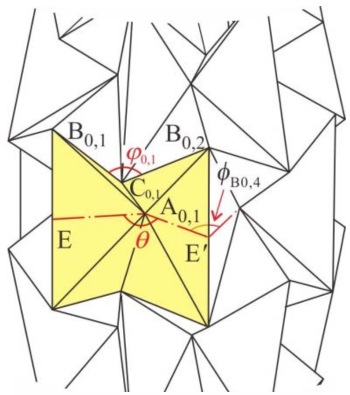
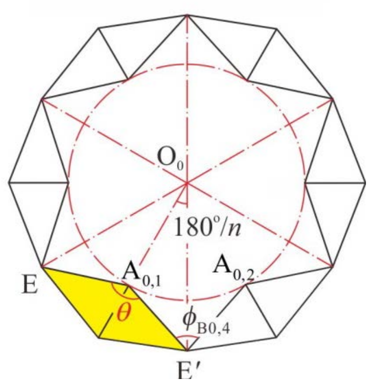
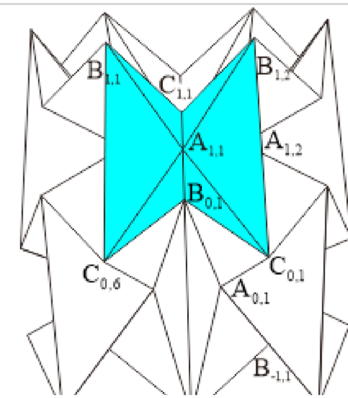
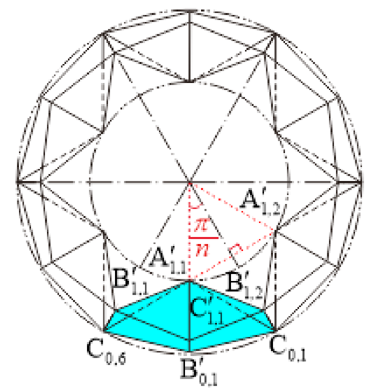


The waterbomb pattern can be treated as consisting of three kinds of linkages, **A**, **B** and **C**, which can be considered as ***plane-symmetric spherical 6R linkages***

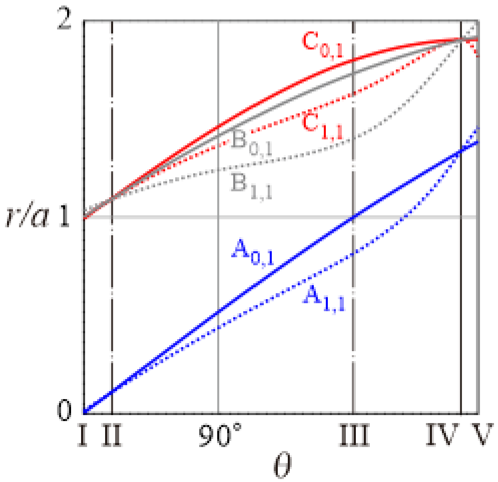


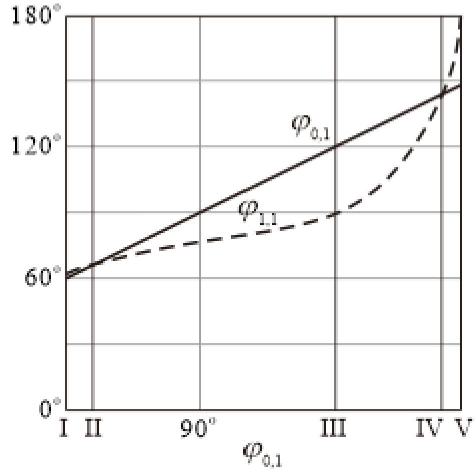
4. Waterbomb origami tube

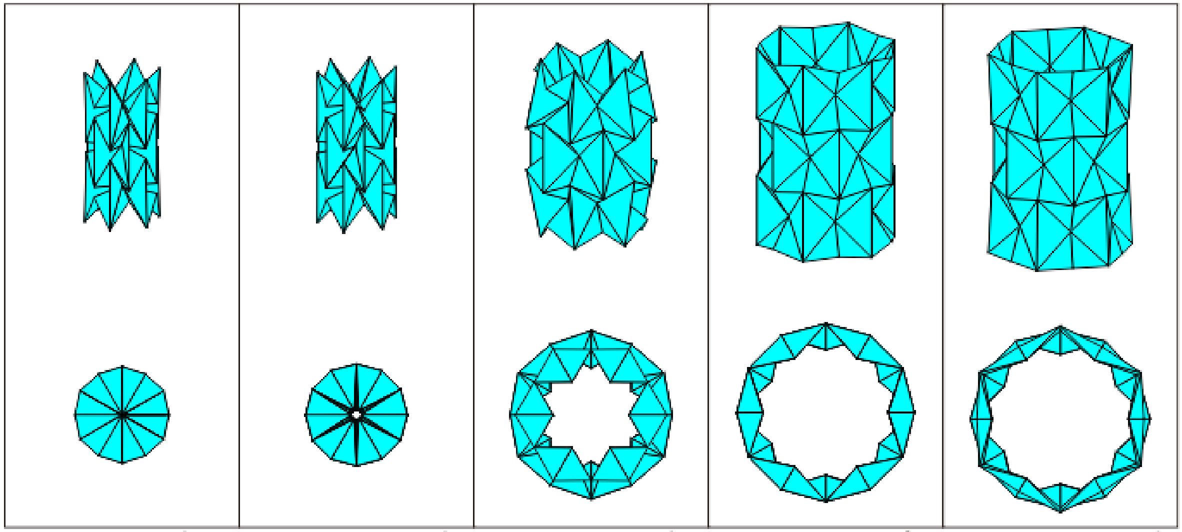
➢ Kinematics of the waterbomb tube

	When *m* is odd	When *m* is even
3D view and projection	 	 
constraints	1. to form the cylindrical tessellation $\frac{\phi_{B0,4}}{2}+\frac{180^\circ}{n}=\frac{\varphi_{0,1}}{2}$ 2. linkage **A** in the mid row 0 is line and plane symmetric	1. to form the cylindrical tessellation $\cos\phi_{B0,4}=1-\frac{2r_{A1}^2\sin^2\frac{\pi}{n}}{a^2}$ 2. linkages $\mathbf{B_0}$ and $\mathbf{C_0}$ are identical

➢ Kinematics of the waterbomb tube







I: $\theta = 60^\circ$ II: $\theta = 65.88^\circ$ III: $\theta = 120^\circ$ IV: $\theta = 144^\circ$ V: $\theta = 147.96^\circ$

I: the most compact-folding cylindrical configuration

$$\varphi_{0,1\min} = \frac{360^\circ}{n}$$

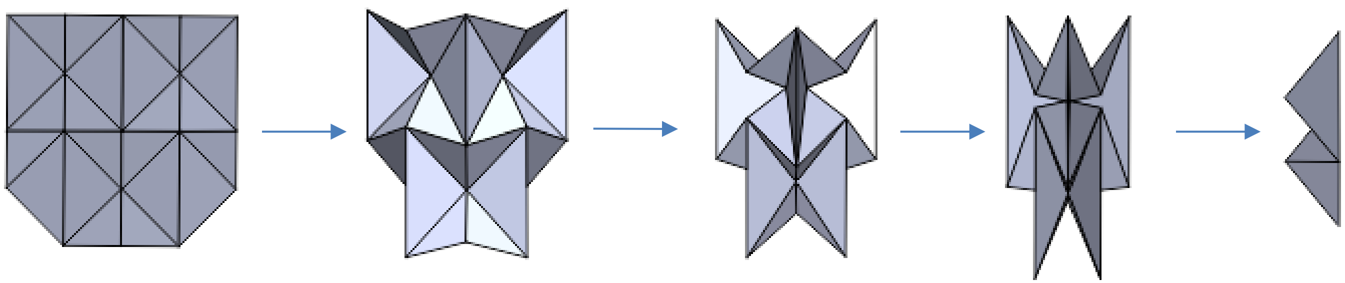
V: the most deployed configuration

$$\varphi_{(m-1)/2,1} = 180^\circ \longrightarrow \varphi_{0,1\max}$$

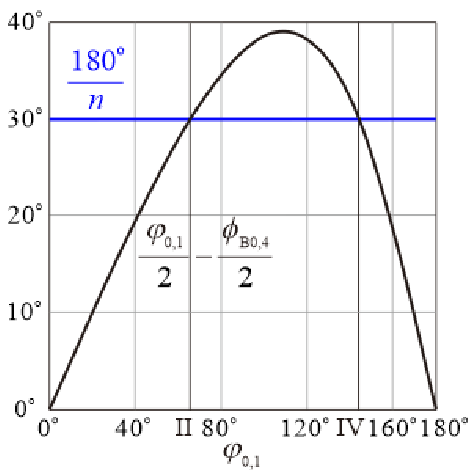
II & IV: the configuration of cylinder with uniform radius

➢ Kinematics of the waterbomb tube

Folding from a flat paper to the waterbomb cylinder:

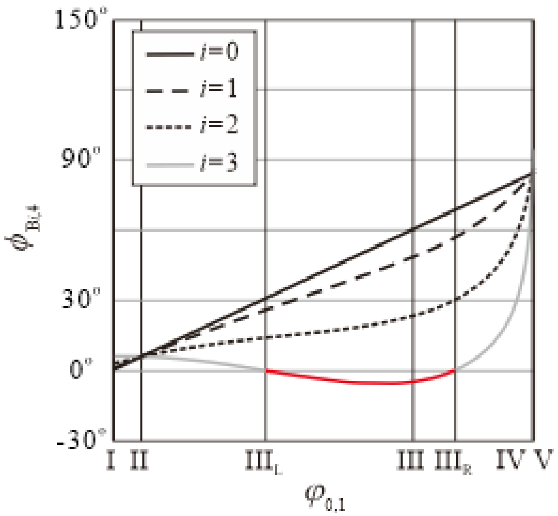


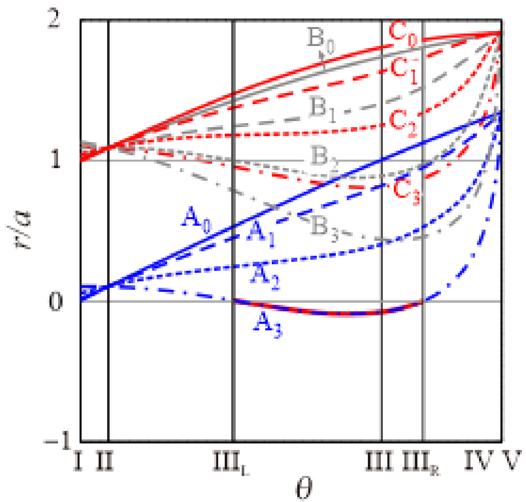
$$\tan\frac{\phi_{B0,4}}{2}=\frac{\tan\frac{\varphi_{0,1}}{2}\left(\cos^4\alpha\tan^4\frac{\varphi_{0,1}}{2}-2\cos2\alpha\cos^2\alpha\tan^2\frac{\varphi_{0,1}}{2}-\cos2\alpha(\cos2\alpha+2)\right)}{3\cos^4\alpha\tan^4\frac{\varphi_{0,1}}{2}+2\cos^2\alpha\tan^2\frac{\varphi_{0,1}}{2}-\cos^2 2\alpha}$$

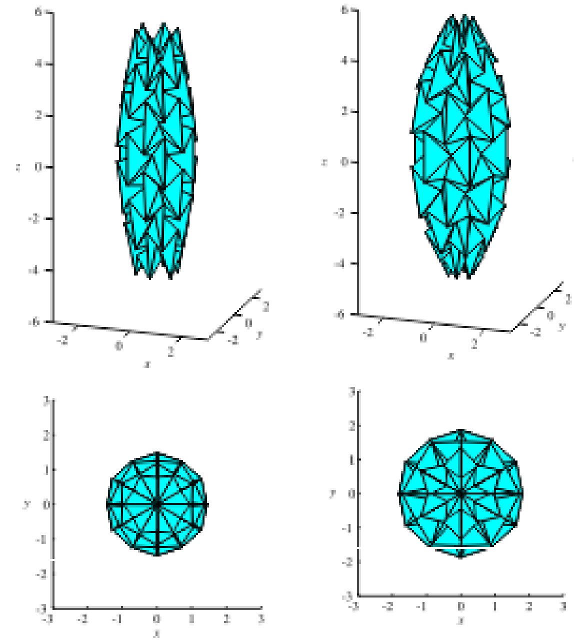


$$\frac{\varphi_{0,1}}{2}-\frac{\phi_{B0,4}}{2}=\frac{180°}{n}$$ → the configuration of cylinder with uniform radius

➢ Kinematics of the waterbomb tube







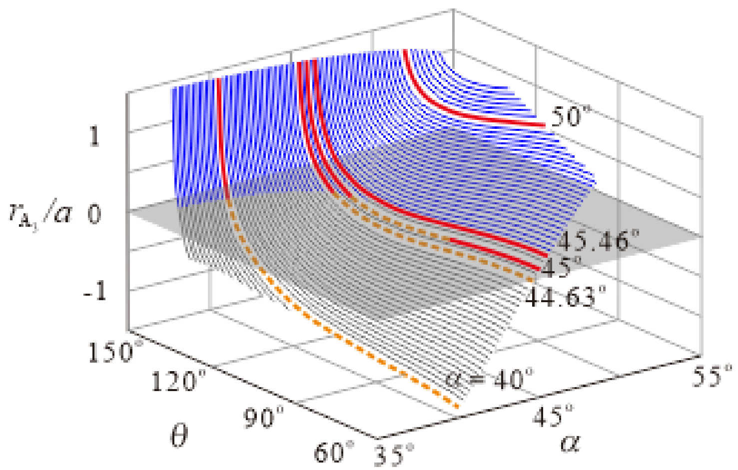
III_L: $\theta = 90.72°$ III_R: $\theta = 128.52°$

Another two special points —— the interference at two ends of the tube

$$r_{\mathrm{A}i} = 0 \longrightarrow \phi_{\mathrm{B}i,4} = 0$$

➢ Kinematics of the waterbomb tube

- The waterbomb tube with varying α :



a. The mechanism-structure-mechanism transition is only observed within

$$44.63^{\circ} \leq \alpha \leq 45.46^{\circ}$$

b. In the range that $45.46^{\circ} < \alpha < 48.62^{\circ}$, the minimum of θ remains constant while the maximum decreases with the increase of α , making the range of rigid motion shrink.

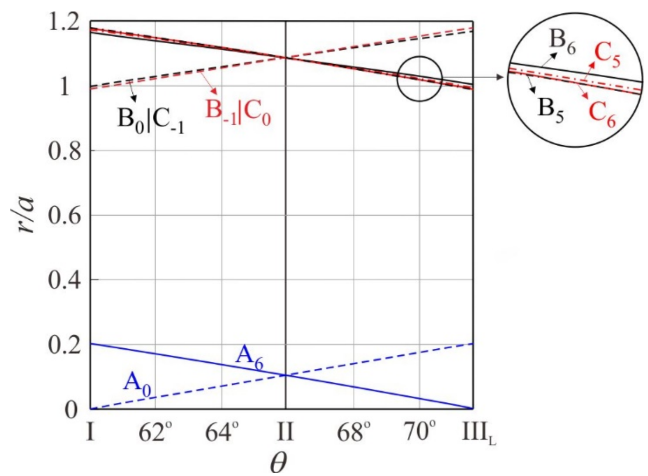
c. In the range that $48.62^{\circ} \leq \alpha \leq 51.4^{\circ}$, the minimum of θ decreases as well as the maximum with the increase of α , so does the range of rigid motion.

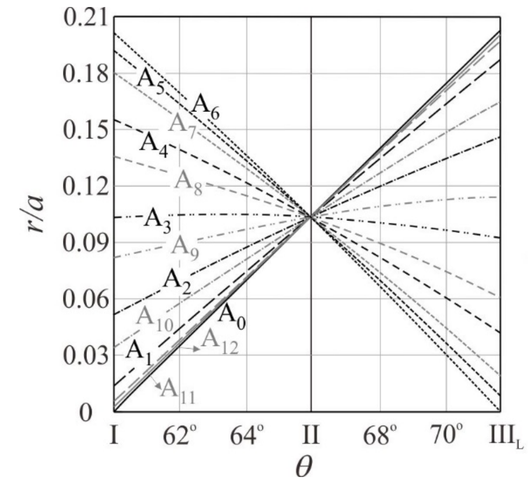
d. When $\alpha < 44.63^{\circ}$, the minimum of remains constant while the maximum θ decreases with the decrease of α , making the range of rigid motion shrink.

➢ Waterbomb wave

- The waterbomb tube with increasing m —— waterbomb wave

The condition with wave： $\theta_{\mathrm{I}} < \theta_{\mathrm{II}} < \theta_{\mathrm{III_L}}$





$n = 6 \quad m = 25 \quad \alpha = 45^{\circ}$

4. Waterbomb origami tube

➢ Waterbomb wave

waterbomb wave when $\alpha = 45^o$

n	θ_I	θ_{II}	θ_{III_L}	m (with interference)	range
5	72º	85.98º	100.6º	33	28.6º
6	60º	65.94º	71.64º	13	11.64º
7	51.43º	54.74º	57.94º	15	6.51º
8	45º	47.07º	49.09º	17	4.09º
9	40º	41.40º	42.76º	19	2.76º
10	36º	36.99º	37.96º	21	1.96º
11	32.73º	33.46º	34.17º	23	1.44º
12	30º	30.55º	31.09º	25	1.09º
13	27.70º	28.12º	28.54º	27	0.84º
14	25.72º	26.06º	26.39º	29	0.67º
15	24º	24.28º	24.54º	31	0.54º
16	22.5º	22.73º	22.94º	33	0.44º
17	21.18º	21.36º	21.54º	35	0.36º
18	20º	20.16º	20.31º	37	0.31º
19	18.95º	19.08º	19.21º	39	0.26º
20	18º	18.11º	18.22º	39	0.22º

The range of the wave shrinks with the increase of *n*

4. Waterbomb origami tube

➢ Waterbomb wave

waterbomb wave when $\alpha < 45^o$

n	α	θ_I	θ_{II}	θ_{III_L}	m (with interference)	range
5	44.7°	72°	81.32°	90.6°	33	18.6°
	44.5°		77.96°	84.06°	35	12.06°
	44.3°		74.36°	76.67°	35	4.67°
	44.18°		72.04°	72°	/	/
6	44.9°	60°	64.38°	68.52°	13	8.52°
	44.8°		62.65°	65.25°	13	5.25°
	44.7°		60.92°	61.74°	39	1.74°
	44.65°		60.06°	60.03°	/	/
7	44.9°	51.43°	52.94°	54.32°	15	2.89°
	44.85°		51.94°	52.40°	15	0.97°
	44.83°		51.48°	51.43°	/	/

a. The values of θ_{II} and θ_{III_L} reduce with the decrease of α, so does the range of wave.

b. The range of α where there exists wave decreases with the increase of *n*.

4. Waterbomb origami tube

➢ Waterbomb wave

waterbomb wave when $\alpha > 45^o$

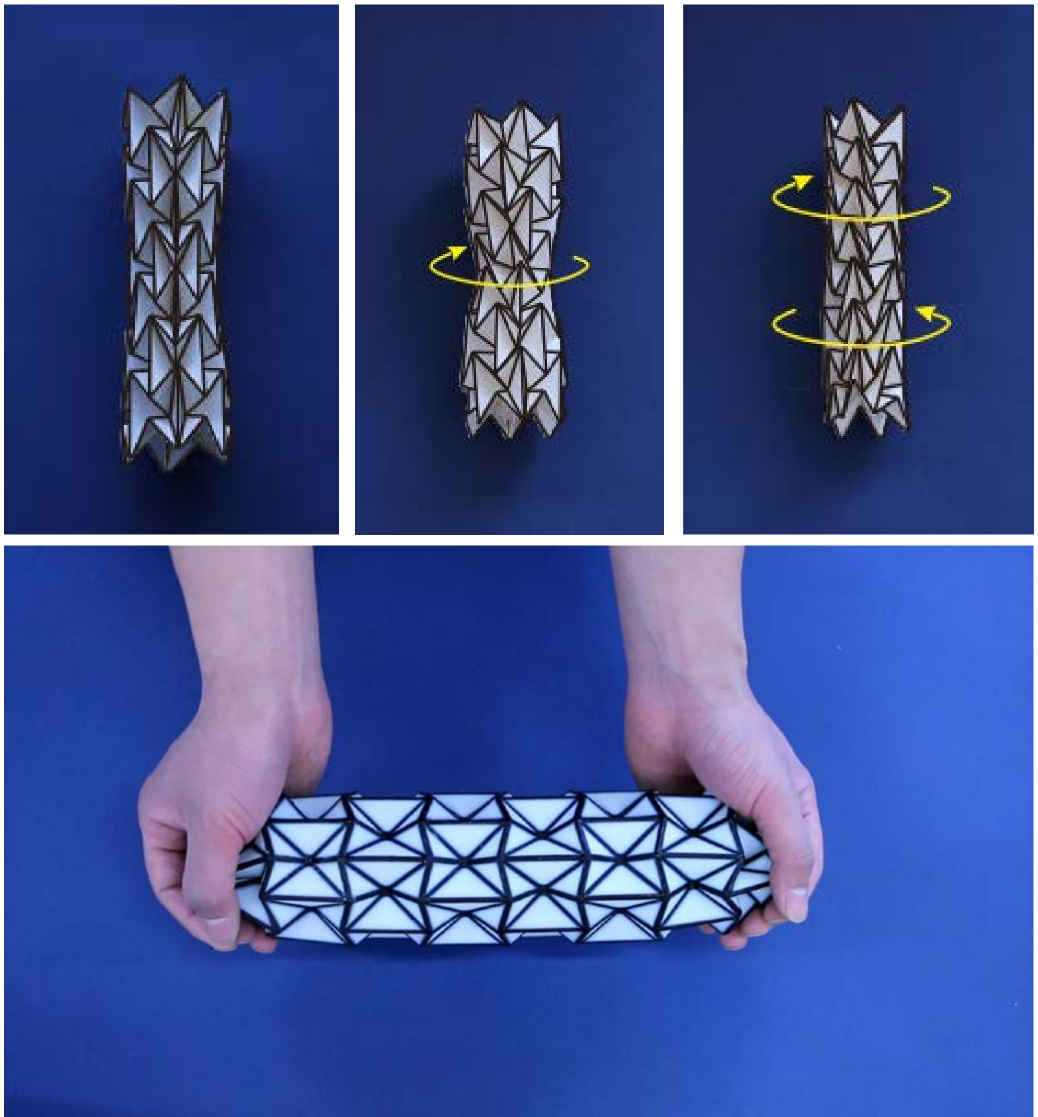
n	α	θ_I	θ_{II}	θ_{III_L}	m (with interference)	range
5	45.2°	72°	88.98°	107.97°	33	35.97°
	45.4°		91.90°	117.48°	13	45.48°
	45.5°		93.31°	128.37°	no interference	56.37°
	45.6°	74.92°	94.80°	127.98°	no interference	53.06°
	45.8°	80.71°	97.71°	127.11°	no interference	46.40°
	46.68°	113.68°	115.58°	117.41°	no interference	115.07° - 116.34°
6	45.5°	60°	73.00°	85.27°	33	25.27°
	46°		79.11°	97.28°	11	37.28°
	46.5°		84.50°	108.97°	11	48.97°
	47.335°		92.60°	141.30°	no interference	81.30 °
	47.343°	60.13°	92.66°	141.33°	no interference	81.20 °
	48°	72.39°	98.52°	140°	no interference	67.61 °
	49°	88.75°	106.97°	139.50°	no interference	50.75 °
	50.90°	127°	128.17°	129.44°	no interference	128.04° -128.64°

a. The value of θ_{II} increases with the increase of α, while θ_{III_L} and the range increase first and then decrease.

b. The range of α where there exists wave increases with the increase of n.

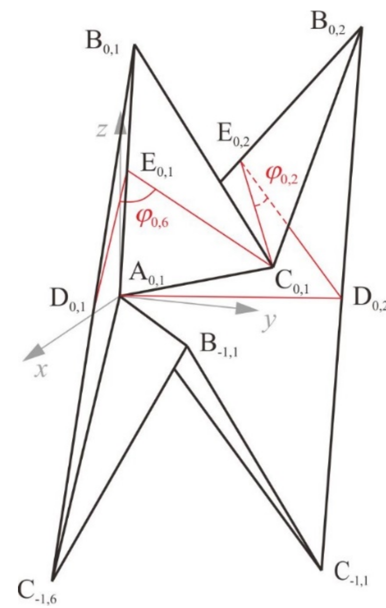
4. Waterbomb origami tube

➢ Twist of waterbomb tube



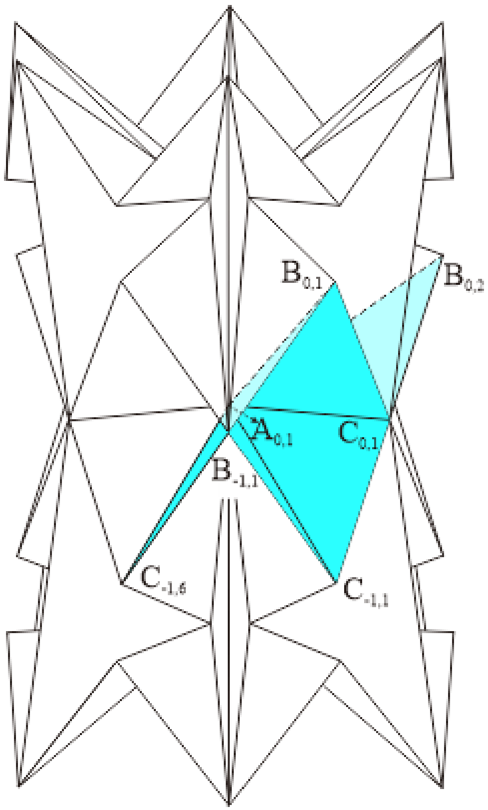
Question:
Is the twist motion rigid and when the twist begins ?

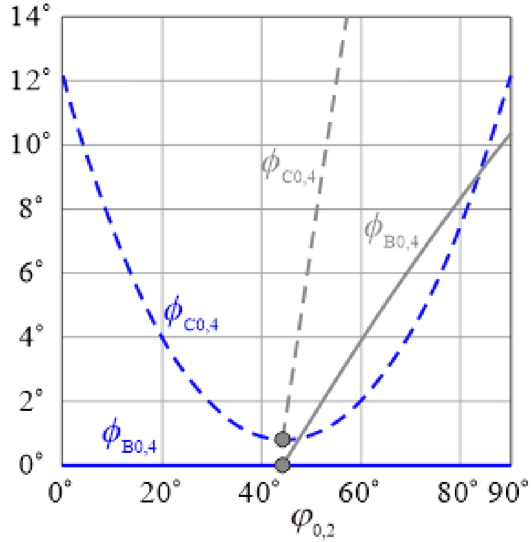
Answer:
The mid row of the cylinder is assumed to be fully squeezed when twist begins.



4. Waterbomb origami tube

➢ Twist of waterbomb tube





Conclusions:

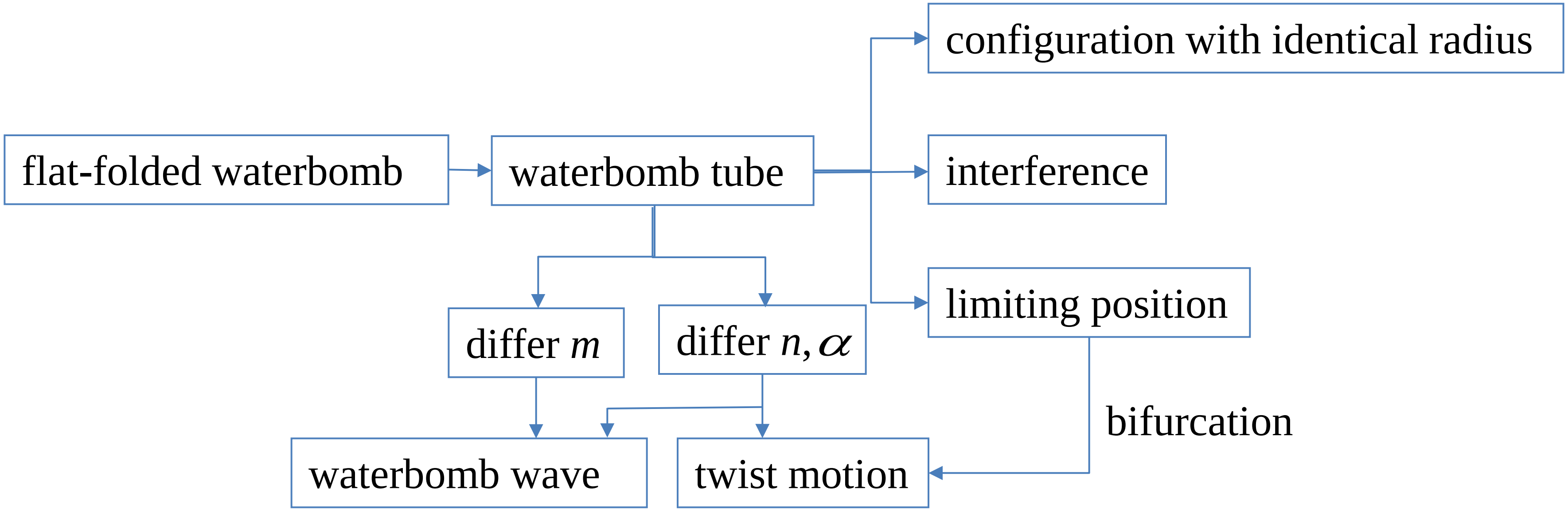
a.the twist of the mid row is a rigid motion

b.the subsequent twist of neighbouring rows cannot occur without material deformation

c.The configuration where the mid row is fully squeezed is a bifurcation position.

4. Waterbomb origami tube

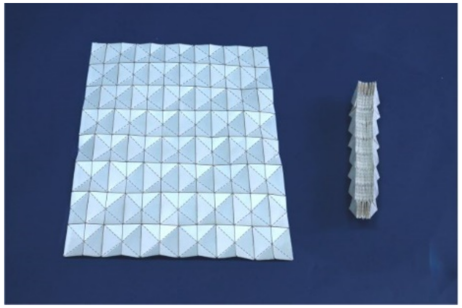
➢ **Summary**



Outline

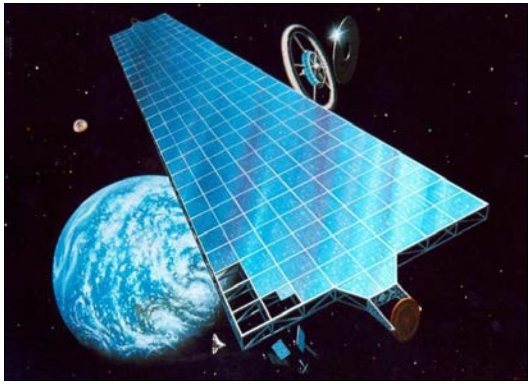
1. Introduction
2. Symmetric folding of the waterbomb origami pattern
3. Thick panel problem of the waterbomb pattern
4. Waterbomb origami tube
5. Perspectives on the waterbomb pattern

5. Perspectives on the waterbomb pattern

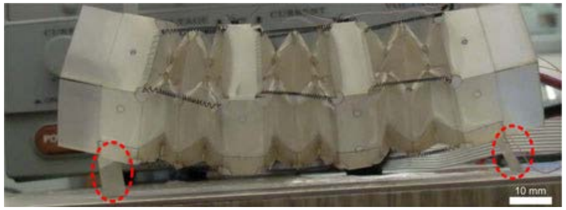


flat-foldable surface

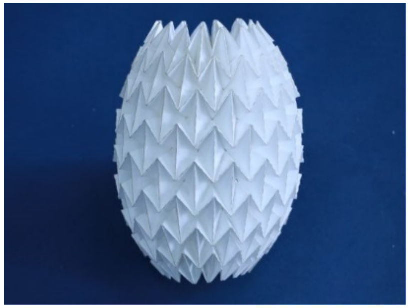
solar panels



space mirror



transformable worm robot



deformable tube

medical stent graft

deformable robot wheel

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Thank you for your attention！

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